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ON THE INTERSECTION OF EDGES OF A GEOMETRIC GRAPH BY STRAIGHT LINES

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A geometric graph (= gg) is a pair $G = \langle V, E\rangle$, where V is a finite set of points (= vertices) in general position in the plane, and E is a set of open straight line segments (= edges) whose endpoints are in V. G is a convex gg (= cgg) if V is the set of vertices of a convex polygon. For $n \geq 1$, $0 \leq e \leq \binom{n}{2}$ and $m \geq 1$ let $I = I(n, e, m)$ ($I_c = I_c(n, e, m)$) be the maximal number such that for every gg (cgg) G with n vertices and e edges there exists a set of m lines whose union intersects at least I (I_c) edges of G. In this paper we determine $I_c(n, e, m)$ precisely for all admissible n, e and m and show that $I(n, e, m) = I_c(n, e, m)$ if $2me \geq n^2$ and in many other cases.

1. Introduction

A *geometric graph* (= gg) is a pair $G = \langle V, E\rangle$, where V is a finite set of points (= vertices) in general position in the plane, and E is a set of open straight line segments (= edges) whose endpoints are in V. G is a convex gg (= cgg) if V is the set of vertices of a convex polygon (or if $|V| \leq 2$). For $S \subset R^2$, denote by $I(G, S)$ the number of edges of G that intersect S. Denote by $I(G, m)$ the maximum of $I(G, M)$, where M rages over all unions of m straight lines in R^2. (One can easily check that this maximum is attained for some m lines in $R^2 \setminus V$, since the edges are open line segments.) For $n \geq 1$, $0 \leq e \leq \binom{n}{2}$ and $m \geq 1$ define

$$I(n, e, m) = \min\{I(G, m)\colon G \text{ is a gg with } n \text{ vertices and } e \text{ edges}\}, \qquad (1.1)$$

and

$$I_c(n, e, m) = \min\{I(G, m)\colon G \text{ is a cgg with } n \text{ vertices and } e \text{ edges}\}. \qquad (1.2)$$

In this paper we investigate the functions $I(n, e, m)$ and $I_c(n, e, m)$. In Section 5 we prove that $I_c(n, e, m) = h(n, e, m)$, where h is the function defined in (1.3) below. We conjecture that $I(n, e, m) = I_c(n, e, m)$ ($= h(n, e, m)$) for all admissible values of n, e and m. In Section 4 we prove this conjecture for all values of n, e, m that satisfy $2me \geq n^2$ and for many other values. The problem of determining or estimating $I(n, e, 1)$ was raised by Erdös, Lovász, Simmons and Straus in [2]. They conjectured that $I(n, cn + 1, 1) \geq c^2$. We show that $I(n, cn + 1, 1) = c^2 + c + 1$ provided $n \leq [n/2c] \cdot (2c + 2)$, which partially settles this conjecture.

Definition of $h(n, e, m)$ ($n, m \geq 1$, $0 \leq e \leq \binom{n}{2}$).
(All numbers appearing below are integers.)

Suppose $n = 2mv - \rho \quad (0 \leq \rho < 2m)$, i.e., $v = \lceil n/2m \rceil$,
$e = n \cdot (k-1) + s \quad (0 \leq s < n)$
and $sk = n \cdot t + r \quad (0 \leq r < n)$.

$$\begin{cases} \text{If } k < v \text{ (i.e., } 2mk < n\text{) then} \\ h(n, e, m) = m \cdot k(k-1) + 2mt + \min(2m, \lceil r/k \rceil). \\ \text{If } k \geq v \text{ (i.e., } 2mk \geq n\text{) then} \\ h(n, e, m) = e - (v-1)(n - mv) \ (= e - \rho\binom{v-1}{2} - (2m-\rho)\binom{v}{2}). \end{cases} \tag{1.3}$$

For $m = 1$ one can easily check that $h(n, e, 1) = k \cdot (k-1) + 2t + \min(2, \lceil r/k \rceil)$ for *all* $0 \leq e \leq \binom{n}{2}$.

Remark. The function $h(n, e, m)$ can be approximated by a simple function of n, e and m as follows:
Put

$$\bar{h}(n, e, m) = \begin{cases} m(e^2/n^2 + e/n) & \text{if } e \leq n^2/(2m) \\ e - n^2/(4m) + n/2 & \text{if } 3 \geq n^2/(2m). \end{cases}$$

Then $|h(n, e, m) - \bar{h}(n, e, m)| \leq 2.25m$ for all admissible values of n, e and m. (In fact, if $e < n^2/(2m)$ then $\bar{h} - 2m \leq h \leq \bar{h} + 2.25m$ and if $e \geq n^2/(2m)$ then $\bar{h} - 0.25m \leq h \leq \bar{h}$.) The verification of these estimates is left to the reader.

Some of our results follow from the following interesting geometric lemma, proved in Section 4.

Lemma. *Let V be a set of n points in general position in the plane and let $n = n_1 + n_2 + \cdots + n_{2m}$ be a decomposition of n into $2m$ nonnegative integers. Then there exist m lines $l_1, l_2, \ldots, l_m \subset R^2 \setminus V$ and a partition of V into $2m$ pairwise disjoint subsets $V_1, V_2, \ldots, V_{2m}$, such that $|V_i| = n_i$ and every two distinct subsets V_i, V_j are separated by at least one of the m lines.*

2. General properties of $I(G, m)$

The following two observations are immediate consequences of the definitions.

Observtion 2.1. $I(G, m_1) \leq I(G, m_1 + m_2) \leq I(G, m_1) + I(G, m_2)$.

Observation 2.2. If $G_1 = \langle V, E_1 \rangle$, $G_2 = \langle V, E_2 \rangle$ and $G = \langle V, E_1 \cup E_2 \rangle$ are geometric graphs on the same set of vertices, then

$$I(G_1, m) \leq I(G, m) \leq I(G_1, m) + I(G_2, m).$$

The next observation is used in Section 4 to obtain lower bounds for $I(n, e, m)$.

Observation 2.3. If $G = \langle V, E \rangle$ is a gg and $m \leq p$, then

$$I(G, m) \geq (m/p) I(G, p).$$

Proof. Let $P = \{l_1, l_2, \ldots, l_p\}$ be a set of p lines such that $I(G, \bigcup P) = I(G, p)$. Let T be the set of all ordered pairs (f, M), where f is an edge of G, M is a subset of P of cardinality m and at least one member of M intersects f.

Clearly there are $I(G, p)$ edges f of G that appear as a first coordinate of an element of T, and each such edge appears in at least $\binom{p-1}{m-1}$ elements of T. On the other hand, every set M of m lines appears in at most $I(G, m)$ elements of T. Therefore

$$\binom{p-1}{m-1} \cdot I(G, p) \leq |T| \leq \binom{p}{m} \cdot I(G, m),$$

which implies the desired results. $\square$

3. The extremal examples

In this section we obtain an upper bound for the function $I_c(n, e, m)$ (which is, of course, also an upper bound for $I(n, e, m)$). As we shall see in Sections 4 and 5, this bound is actually the exact value of $I_c(n, e, m)$ for all possible values of n, e and m, and it equals $I(n, e, m)$ in many cases.

Recall the definition of the function $h(n, e, m)$ given in (1.3).

Theorem 3.1. *For all possible n, e and m ($I(n, e, m) \leq$) $I_c(n, e, m) \leq h(n, e, m)$.*

Proof. We prove the theorem by constructing for any given n and e a cgg G with n vertices and e edges such that $I(G, m) \leq h(n, e, m)$ for all m. We first describe the examples and then estimate $I(G, m)$ for each such example G.

Let $v_0, v_1, \ldots, v_{n-1}, v_n = v_0$ be the vertices of a convex polygon P, and assume they appear in this cyclic order on its boundary. Put $V = \{v_0, v_1, \ldots, v_{n-1}\}$.

We set out to define a linear order on the edges of the (abstract) complete graph K on V, as follows.

We say that an edge $v_i v_j$ of K has *length* $d = d(v_i v_j)$ $(1 \leq d \leq \frac{1}{2}n)$ if $i \equiv j + d \pmod{n}$ or $j \equiv i + d \pmod{n}$. Denote by E_d the set of edges of length d, and put $G_d = \langle V, E_d \rangle (1 \leq d \leq \frac{1}{2}n)$. The (abstract) graph G_d has $c = \gcd(n, d)$ connected components $C_{d,0}, \ldots, C_{d,c-1}$, where $v_i \in C_{d,i}$ for $0 \leq i < c$. If $d < \frac{1}{2}n$ then each $C_{d,i}$ is a cycle of length n/c, and if n is even and $d = \frac{1}{2}n$ then each $C_{d,i}$ is an isolated edge. We order the edges of K according to the following rules:

I. Short edges precede long ones.

II. If $0 \leqslant i < j < \gcd(n, d)$ then the edges of $C_{d,i}$ precede those of $C_{d,j}$. (The particular chosen ordering of the components $C_{d,i}$ is just a matter of convenience.)

III. The edges of $C_{d,i}$ are ordered as follows: $v_i v_{i+d}, v_{i+d} v_{i+2d}, \ldots, v_{i+(t-1)d} v_i$, where $t = n/\gcd(n, d)$ and all subscripts are reduced modulo n.

For $0 \leqslant e \leqslant \binom{n}{2}$ let $G(e)$ be the cgg on V whose edges are the first e edges according to the linear order defined above. Our aim is to show that

$$I(G(e), m) \leqslant h(n, e, m), \quad \text{for all } m \geqslant 1. \tag{3.1}$$

Before doing that, however, we introduce some auxiliary notions, related to cgg's on V. These will be useful here, in the proof of Theorem 3.1, and also later, in Section 5, where we determine the exact value of $I_c(n, e, m)$.

Recall that $V = \{v_0, v_1, \ldots, v_{n-1}\}$ and the points $v_0, v_1, \ldots, v_{n-1}, v_n = v_0$ appear in this cyclic order on the boundary of the convex polygon $P = \operatorname{conv} V$. For $0 \leqslant i < n$ let a_i be the segment joining v_i to v_{i+1}. Define $A = \{a_0, a_1, \ldots, a_{n-1}\}$. A is just the set of edges of P.

In what follows, addition of subscripts is always reduced modulo n. Every edge b on V can be written uniquely as $b = v_i v_{i+d}$, where $0 \leqslant i < n$ and $1 \leqslant d < \frac{1}{2}n$, or $0 \leqslant i < \frac{1}{2}n$ and $d = \frac{1}{2}n$ (if n is even). The number d is just the length of b. Define $W(b) = \{a_i, a_{i+1}, \ldots, a_{i+d-1}\}$. (Note that $|W(b)| = d$, and if $b \in A$ then $W(b) = \{b\}$; $W(b)$ is called the *weak side* of b. Note also that for edges of length $\frac{1}{2}n$ our definition of $W(b)$ depends on the particular numbering $v_0, v_1, \ldots, v_{n-1}$ of V.)

For $a \in A$ define

$$f_b(a) = \begin{cases} 1 & \text{if } a \in W(b) \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we define, for any cgg $H = \langle V, E(H) \rangle$ on V, a function $f_H: A \to Z^+$ as follows:

$$f_H(a) = \sum \{f_b(a): b \in E(H)\} \quad (= |\{b \in E(H): a \in W(b)\}|).$$

Note that $\sum \{f_H(a): a \in A\}$ is just the sum of lengths of the edges of H. Note also that if b is an edge of H, and l is a line in $R^2 \setminus V$ that intersects the boundary of $P = \operatorname{conv} V$ in a_i and in a_j, then l intersects b iff a_i and a_j lie on different sides of b, i.e., iff $f_b(a_i) + f_b(a_j) = 1$. Therefore the total number of edges of H that intersect l is at most $f_H(a_i) + f_H(a_j)$.

This last observation can be sharpened as follows: We call a line l in $R^2 \setminus V$ of type $a_i a_j$ and write $l = a_i a_j$, if it intersects the boundary of P in a_i and a_j. We assign to such a line a *length* $d = d(l) = d(a_i a_j)$, $1 \leqslant d \leqslant \frac{1}{2}n$, if $j \equiv i \pm d \pmod{n}$. Note that if $l = a_i a_j$ and b is an edge on V, then $f_b(a_i) + f_b(a_j) \leqslant 1$ unless $d(l) < d(b)$ (and $a_i, a_j \in W(b)$). These observations clearly imply

Proposition 3.2.

(i) *If* $l = a_i a_j$, *then*

$$I(H, l) \leqslant f_H(a_i) + f_H(a_j).$$

(ii) *If* $l = a_i a_j$ *and* $d(l) \geqslant d(b)$ *for all* $b \in E(H)$, *then*

$$I(H, l) = f_H(a_i) + f_H(a_j).$$

We complete the proof of Theorem 3.1 by proving inequality (3.1).

Let v, ρ, k, s, t and r be as in (1.3). Note that $G(e)$ contains all edges of length $<k$ and s edges of length k. We consider two possible cases.

Case 1. $n > 2km$.

Here all edges of $G = G(e)$ are of length $\leqslant k < \frac{1}{2}n$. It is easily checked that in this case the function f_G is almost constant, i.e., $[\lambda] \leqslant f_G(a) \leqslant \lceil \lambda \rceil$ for all $a \in A$, where λ is the sum of the lengths of the edges of G divided by n. Therefore $f_G(a) \leqslant c + 1$ for all $a \in A$, where $c = 1 + 2 + \cdots + (k-1) + t = \binom{k}{2} + t$. This and part (i) of Proposition 3.2 show that every line l intersects at most $2(c+1)$ edges of G, and thus m lines intersect at most $2m(c+1)$ edges of G. This completes the proof in case $\lceil r/k \rceil \geqslant 2m$. If $\lceil r/k \rceil < 2m$, define $e' = e - \lceil r/k \rceil$, $G' = G(e')$. The sum of the lengths of the edges of G' is $\leqslant nc$. Repeating the same argument we find that $f_{G'}(a) \leqslant c$ for all $a \in A$, and thus m lines intersect at most $2mc$ edges of G'. Since G has only $\lceil r/k \rceil$ additional edges, we conclude that $I(G(e), m) \leqslant 2mc + \lceil r/k \rceil = h(n, e, m)$. This completes the proof of Case 1.

Case 2. $n \leqslant 2km$, i.e., $v \leqslant k$.

In this case $G = G(e)$ contains all edges of length $<v$. Let M be a set of m lines in $R^2 \backslash V$. We must show that M misses at least $\rho\binom{v-1}{2} + (2m-\rho)\binom{v}{2}$ $(= e - h(n, e, m))$ edges of G. M decomposes the boundary of $P = \operatorname{conv} V$ into μ $(\leqslant 2m)$ pairwise disjoint (open) arcs $A_1, A_2, \ldots, A_\mu$. If $\mu < 2m$ put $A_i = \emptyset$ for $\mu < i \leqslant 2m$. Define, for $1 \leqslant i \leqslant 2m$, $\gamma_i = |V \cap A_i|$. Clearly M misses every edge of G that joins two vertices in the same arc. Thus, in an arc that contains γ points of V, M misses at least $g(\gamma)$ edges, where

$$g(\gamma) = \begin{cases} \dbinom{\gamma}{2} & \text{if } \gamma \leqslant v \\[2ex] \dbinom{v}{2} + \alpha(v-1) & \text{if } \gamma = v + \alpha,\ \alpha \geqslant -1. \end{cases}$$

Altogether M misses at least $\sum_{i=1}^{2m} g(\gamma_i)$ edges of G, and $\sum_{i=1}^{2m} \gamma_i = n = 2vm - \rho$. Since g is the restriction to the nonnegative integers of a real convex function $\bar{g}$ (say, $\bar{g}(x) = \frac{1}{2}x(x-1)$ for $x \leqslant v-1$, $\bar{g}(x) = (v-1)(x - \frac{1}{2}v)$ for $x \geqslant v-1$), it follows that the minimum of $g(\gamma_1) + \cdots + g(\gamma_{2m})$ over all $2m$-tuples $(\gamma_1, \ldots, \gamma_{2m})$ of nonnegative integers with sum $2vm - \rho$ is attained when the numbers γ_i are as equal as possible, i.e., when $\gamma_1 = \gamma_2 = \cdots = \gamma_\rho = v - 1$, $\gamma_{\rho+1} =$

$\cdots = \gamma_{2m} = \nu$. It follows that

$$\sum_{i=1}^{2m} g(\gamma_i) \geq \rho\binom{\nu-1}{2} + (2m-\rho)\binom{\nu}{2}.$$

We conclude that in Case 2 $I(G, m) \leq h(n, e, m)$. This completes the proof of the theorem. □

4. A geometric lemma and its consequences

In this section we prove the geometric lemma mentioned in Section 1, and apply it to obtain a lower bound for $I(n, e, m)$.

Lemma 4.1. *Let V be a set of n points in general position in the plane and suppose $n = \sum_{i=1}^{2m} n_i$, where n_i are nonnegative integers. Then there exist m lines $l_1, l_2, \ldots, l_m \subset R^2 \setminus V$ and a partition of V into $2m$ pairwise disjoint subsets $V_1, V_2, \ldots, V_{2m}$, such that $|V_i| = n_i$ and every two distinct subsets V_i, V_j are separated by at least one of the m lines.*

In order to prove our lemma we need some further notation and another lemma.

If V is a set of points in the plane and l is a directed line, denote by $N^+(V, l)$ and $N^-(V, l)$ the intersections of V with the open half plane to the right of l and to the left of l, respectively.

Lemma 4.2. *Let a, b, c, d be nonnegative integers. Let V be a set of $a+b+c+d$ points in general position in the plane and let l be a directed line that misses V. Suppose $|N^+(V, l)| = a + b$ and $|N^-(V, l)| = c + d$. Then there exists a directed line l' that misses V such that*

$$|N^+(V, l) \cap N^-(V, l')| = b \quad \text{and} \quad |N^-(V, l) \cap N^-(V, l')| = d. \tag{4.1}$$

Note that Lemma 4.2 is essentially the case $m = 2$ of Lemma 4.1.

Proof. *For* $0 < \alpha < \pi$ consider the projection P_α of R^2 onto l along a line that makes an angle α with l. Call α *critical* if the restriction of P_α to V is not 1–1. If α is not critical, then the natural ordering of $P_\alpha V$ along l induces a linear ordering O_α on V. We make the following observations.

(1) If α is sufficiently close to 0, then every point of $N^-(V, l)$ precedes every point of $N^+(V, l)$ according to O_α.

(2) *If* α is sufficiently close to π, then every point of $N^+(V, l)$ precedes every point of $N^-(V, l)$ according to O_α.

(3) If $0 < \alpha < \beta < \pi$, and there is no critical angle in the closed interval $[\alpha, \beta]$, then $O_\alpha = O_\beta$.

(4) If α, β are not critical and there is just one critical angle between α and β, then O_β is obtained from O_α by transposing one or more disjoint pairs of adjacent elements.

For a non-critical angle α $(0<\alpha<\pi)$, denote by $f(\alpha)$ the number of points of $N^+(V, l)$ among the first $b+d$ points with respect to O_α. By observations 1 and 2, if α is close to 0 then $f(\alpha)=\max(0, b-c)\leqslant b$, and if α is close to π then $f(\alpha)=b+\min(a, d)\geqslant b$. By Observations 3 and 4, $f(\alpha)$ changes by at most 1 as α passes through a critical angle. Thus there exists some $\bar{\alpha}$, $0<\bar{\alpha}<\pi$, for which $f(\bar{\alpha})=b$. Let l' be a directed line in this direction that satisfies $|N^-(V, l')|=b+d$. (There exists such a line since $\bar{\alpha}$ is not critical.) One can easily check that l' satisfies (4.1). □

Proof of Lemma 4.1. Let $l_1 \subset R^2 \setminus V$ be a directed line such that

$$|N^+(V, l_1)|=\sum_{i=1}^{m} n_{2i-1} \quad \text{and} \quad |N^-(V, l_1)|=\sum_{i=1}^{m} n_{2i}$$

By Lemma 4.2 there exists a directed line $l_2 \subset R^2 \setminus V$ such that

$$V_1=N^+(V, l_1)\cap N^-(V, l_2) \quad \text{and} \quad V_2=N^-(V, l_1)\cap N^-(V, l_2)$$

satisfy $|V_1|=n_1$ and $|V_2|=n_2$.

We continue by induction. Assume we have defined pairwise disjoint sets $V_1, \ldots, V_{2r-1} \subset N^+(V, l_1)$, $V_2, \ldots, V_{2r} \subset N^-(V, l_1)$ of sizes $n_1, \ldots, n_{2r-1}$, $n_2, \ldots, n_{2r}$ respectively $(1\leqslant r\leqslant m-2)$. Put $\bar{V}=V\setminus(V_1\cup V_2\cup\cdots\cup V_{2r-1}\cup V_{2r})$. By Lemma 4.2 there exists a directed line $l_{r+2}\subset R^2\setminus\bar{V}$ such that

$$V_{2r+1}=N^+(\bar{V}, l_1)\cap N^-(\bar{V}, l_{r+2}) \quad \text{and} \quad V_{2r+2}=N^-(\bar{V}, l_1)\cap N^-(\bar{V}, l_{r+2})$$

satisfy $|V_{2r+1}|=n_{2r+1}$ and $|V_{2r+2}|=n_{2r+2}$. By a small perturbation (if necessary) we can ensure that $l_{r+2}\subset R^2\setminus V$. Finally let

$$V_{2m-1}=N^+(V, l_1)\setminus(V_1\cup\cdots\cup V_{2m-3}) \quad \text{and} \quad V_{2m}=N^-(V, l_1)\setminus(V_2\cup\cdots\cup V_{2m-2}).$$

We complete the proof by showing that every two distinct sets V_i, V_j are separated by at least one of the lines. Suppose $1\leqslant i<j\leqslant 2m$. If $i\not\equiv j \pmod 2$ then l_1 separates V_i from V_j. If i and j are even then $l_{(i+2)/2}$ separates V_i from V_j and if i and j are odd then $l_{(i+3)/2}$ separates V_i from V_j. This completes the proof. □

Corollary 4.3. *Let $G=\langle V, E\rangle$ be a gg with n vertices and e edges. If $n=n_1+n_2+\cdots+n_{2m}$, where n_i are nonnegative integers, then*

$$I(G, m)\geqslant e-\sum_{i=1}^{2m}\binom{n_i}{2}. \tag{4.2}$$

In particular, if $n=2mv-\rho$, where $v=\lceil n/(2m)\rceil$, as in (1.3), *we obtain*

$$I(G, m)\geqslant e-\rho\binom{v-1}{2}-(2m-\rho)\binom{v}{2} \quad (=e-(v-1)(n-mv)). \tag{4.3}$$

Proof. Let $l_1, \ldots, l_m$ and $V_1, \ldots, V_{2m}$ be as in Lemma 4.1. Since every edge that joins vertices of different sets V_i intersects at least one of the lines l_j, the union of these m lines misses at most $\sum_{i=1}^{2m} \binom{n_i}{2}$ edges of G. This implies (4.2) (and (4.3)). □

Note that by the convexity of the function $\binom{x}{2}$, the right hand side of (4.2) is maximized by taking the parts n_i as equal as possible, and thus inequality (4.3) implies all the inequalities (4.2).

Combining Corollary 4.3 with Observation 2.3 we obtain the following:

Theorem 4.4. *$I_c(n, e, m) \geq I(n, e, m) \geq (m/p)(e - (\lceil n/(2p) \rceil - 1)(n - p\lceil n/(2p) \rceil))$, for all $p \geq m$.*

As noted in Section 1, we conjecture the following:

Conjecture 4.5. *For all possible values of n, e and m, $I(n, e, m) = h(n, e, m)$.*

Combining Theorem 4.4 with Theorem 3.1 we can prove this conjecture in many cases. The following two theorems cover some of these cases. In what follows $k = [e/n] + 1$, as in (1.3).

Theorem 4.6. *If $2mk \geq n$, then*

$$I(n, e, m) = I_c(n, e, m) = h(n, e, m). \tag{4.4}$$

In particular, $I(n, e, m) = e$ if $m \geq \frac{1}{2}n$.

Proof. By Theorem 3.1 $I(n, e, m) \leq I_c(n, e, m) \leq h(n, e, m)$. Conversely, from (4.3) and (1.3) it follows that $I(G, m) \geq h(n, e, m)$ for all gg's G with n vertices and e edges, i.e., $I(n, e, m) \geq h(n, e, m)$. □

Theorem 4.7.

(1) *Let s, k be as in (1.3), and suppose $2mk < n$. If $n = 2kp$ for some positive integer p, and $s \leq 1$, then*

$$I(n, e, m) = I_c(n, e, m) = h(n, e, m).$$

(2) *If $2pc - p + 1 \leq n \leq 2pc + 3p - 1$ for some positive integers p, c, then*

$$I(n, nc, 1) = h(n, nc, 1) = c^2 + c. \tag{4.5}$$

(*In particular,* (4.5) *holds whenever $n > (2c - 1)\lceil \frac{1}{2}c \rceil$.*).

(3) *If $2pc \leq n \leq 2pc + 2p$ for some positive integers p, c, then*

$$I(n, nc + 1, 1) = h(n, nc + 1, 1) = c^2 + c + 1. \tag{4.6}$$

(*In particular,* (4.6) *holds whenever $n \geq 2c^2$.*)

Proof. The inequalities $I \leqslant I_c \leqslant h$ follow from Theorem 3.1 in all cases. The opposite inequalities $I \geqslant h$ follow from Theorem 4.4. Note that in (2) $\lceil n/(2p) \rceil$ may be either c or $c+1$ or $c+2$, and in (3) $\lceil n/(2p) \rceil$ may be either c or $c+1$. It is convenient to treat these cases separately. □

The next theorem summarizes the known asymptotic bounds for $I(n, e, m)$.

Theorem 4.8. *Put* $I = I(n, e, m)$, $h = h(n, e, m)$.

(1) *If* $n^2/(2m) \leqslant e$, *then*

$$e - n^2/(4m) + \tfrac{1}{2}n - \tfrac{1}{4}m \leqslant I = h \leqslant e - n^2/(4m) + \tfrac{1}{2}n.$$

(2) *If* $n^2/(2m) - n \leqslant e \leqslant n^2/(2m)$, *then*

$$e - n^2/(4m) + \tfrac{1}{2}n - \tfrac{1}{4}m \leqslant I \leqslant h \leqslant e - n^2/(4m) + \tfrac{1}{2}n + \tfrac{9}{4}m.$$

(3) *If* $e \leqslant n^2/(2m) - \tfrac{1}{2}n$, *then*

$$\begin{aligned} & m(e^2/n^2 + e/n + \tfrac{1}{4})\Big(1 + \frac{2e+n}{n^2}\Big)^{-1} - \tfrac{1}{4}m \\ & \leqslant m(e^2/n^2 + e/n + \tfrac{1}{4})\Big(2\Big(1 + \frac{2e+n}{n^2}\Big)^{-1} - \Big(1 + \frac{2e+n}{n^2}\Big)^{-2}\Big) - \tfrac{1}{4}m \\ & \leqslant I \leqslant h \leqslant m(e^2/n^2 + e/n) + \tfrac{9}{4}m. \end{aligned}$$

It follows that for all admissible values of m, $(1 - (m+1)^{-2})h - 2.5m \leqslant I \leqslant h$. Moreover, if n, e and m vary in such a way that $e/n \to \infty$, $e/n^2 \to 0$ and $e \leqslant n^2/2m - \tfrac{1}{2}n$, then

$$\lim \frac{I(n, e, m)}{m(e^2/n^2 + e/n)} = \lim \frac{h(n, e, m)}{m(e^2/n^2 + e/n)} = 1.$$

Proof. (1) follows from Theorem 4.6 and the remark concerning $\bar{h}$ in Section 1. ($n^2/(2m) \leqslant e$ clearly implies $2mk = 2m([e/n] + 1) \geqslant n$.)

Combining the same remark with Theorem 3.1 we conclude that if $e \leqslant n^2/(2m)$, then $I(n, e, m) \leqslant h(n, e, m) \leqslant m(e^2/n^2 + e/n) + 2.25m$. One can easily check that if $n^2/(2m) - n \leqslant e \leqslant n^2/(2m)$ then $m(e^2/n^2 + e/n) \leqslant e - n^2/(4m) + \tfrac{1}{2}n$. This implies the upper bounds for I and h that appear in (2) and (3).

To prove the lower bounds we first show that

$$I(n, e, m) \geqslant m\Big(\frac{2e+n}{2p} - \frac{n^2}{4p^2} - \frac{1}{4}\Big), \quad \text{for all } p \geqslant m. \tag{4.7}$$

By Theorem 4.4

$$I(n, e, m) \geqslant \frac{m}{p}(e - (\lceil n/(2p) \rceil - 1)(n - p\lceil n/(2p) \rceil)), \quad \text{for all } p \geqslant m.$$

Writing $\lceil n/(2p) \rceil = n/(2p) + \varepsilon$, where $0 \leqslant \varepsilon < 1$, we obtain

$$I(n, e, m) \geqslant \frac{m}{p}\left(e - \left(\frac{n}{2p} + \varepsilon - 1\right)(\tfrac{1}{2}n - p\varepsilon)\right) = m\left(\frac{e}{p} - \frac{n^2}{4p^2} + \frac{n}{2p} - \varepsilon(1-\varepsilon)\right)$$

$$\geqslant m\left(\frac{2e+n}{2p} - \frac{n^2}{4p^2} - \frac{1}{4}\right).$$

Substituting $p = m$ in Inequality (4.7) we obtain the lower bound given in (2). To prove the lower bound of (3), put $p = \lceil n^2/(2e+n) \rceil$. Note that since $e \leqslant n^2/(2m) - \frac{1}{2}n$, $\lceil n^2/(2e+n) \rceil \geqslant n^2/(2e+n) \geqslant m$. Substituting in (4.7) $p = n^2/(2e+n) + \delta$, where, $0 \leqslant \delta < 1$, we conclude that

$$I(n, e, m) \geqslant m\left(\frac{(2e+n)^2}{2(n^2+2e\delta+n\delta)} - \frac{n^2(2e+n)^2}{4(n^2+2e\delta+n\delta)^2} - \frac{1}{4}\right)$$

$$= m\frac{(2e+n)^2}{4n^2}\left(2 \cdot \left(1 + \frac{2e\delta+n\delta}{n^2}\right)^{-1} - \left(1 + \frac{2e\delta+n\delta}{n^2}\right)^{-2}\right) - \tfrac{1}{4}m$$

$$\geqslant m\left(\frac{e^2}{n} + \frac{e}{n} + \frac{1}{4}\right)\left(2\left(1 + \frac{2e+n}{n^2}\right)^{-1} - \left(1 + \frac{2e+n}{n^2}\right)^{-2}\right) - \tfrac{1}{4}m$$

$$\geqslant m\left(\frac{e^2}{n^2} + \frac{e}{n} + \frac{1}{4}\right)\left(1 + \frac{2e+n}{n^2}\right)^{-1} - \tfrac{1}{4}m.$$

(The second inequality follows from the fact that the function $2y^{-1} - y^{-2}$ is decreasing for all $y \geqslant 1$.)

This completes the proof. □

Remarks

(1) We can prove Conjecture 4.5 in some cases that do not follow from Theorem 4.4. In particular, we can prove it for $m = 1$ provided $e \leqslant \frac{3}{2}n$, or $n \equiv 1 \pmod 2$ and $e \geqslant \frac{1}{2}n(n-3)$, or $n \equiv 0 \pmod 3$ and $e = \frac{1}{3}n^2 + 1$.

(2) Erdös, Lovász, Simmons and Straus [2, Conjecture 5.4] conjectured that $I(n, nc+1, 1) \geqslant c^2$. Equality (4.6) shows that actually $I(n, nc+1, 1) = c^2 + c + 1$ if $n \leqslant [n/2c](2c+2)$. In particular, this holds whenever $n \geqslant 2c^2$.

(3) The authors of [2] defined $f(n, r)$ as follows:
$f(n, r) = \min\{e: I(n, e, 1) \geqslant r\}$. They noted that $f(n, 1) = 1$, $f(n, 2) = 2$ and $f(n, 3) = n + 1$, and asked for the determination of $f(n, r)$ in other cases. It is not difficult to see that $f(n, 4) = n + 2$. Regarding larger values of r, equalities (4.5) and (4.6) show that

$$f(n, c^2 + c + 1) = nc + 1$$

provided $n \leqslant [n/2c](2c+2)$. In particular, this is true if $n \geqslant 2c^2$.

(4) Let $\mathcal{U}$ be a set of $2p$ points in general position in the plane. The bigraph B

on $\mathcal{U}$ is the geometric graph on $\mathcal{U}$ in which $u, v \in \mathcal{U}$ are joined iff the line through u and v bisects $\mathcal{U} \setminus \{u, v\}$. There are several papers dealing with bigraphs (see [1, 3, 4]), and the best known upper bound for the number of edges of B is $2\sqrt{2}p^{\frac{3}{2}}$ (see [2, 3]).

Combining the method used in the proof of Corollary 4.3 with the lemma of Lovász [3] we can improve this bound to $\sqrt{3}p^{\frac{3}{2}}(1 + o(1))$. (The true bound is probably much smaller. See [2, Conjecture 5.2].)

5. The determination of $I_c(n, e, m)$

In this section we finally determine $I_c(n, e, m)$ for all possible values of n, e and m.

Theorem 5.1. *For evry n, $m \geq 1$ and $0 \leq e \leq \binom{n}{2}$*

$$I_c(n, e, m) = h(n, e, m),$$

where $h(n, e, m)$ is given in (1.3).

In view of Theorems 3.1 and 4.6 we only have to show that

$$\text{if } k = [e/n] + 1 < n/(2m), \text{ then } I_c(n, e, m) \geq h(n, e, m). \tag{5.1}$$

This can be rephrased as follows:

$$\text{Suppose } e = n(k-1) + s \ (0 \leq s < n), \text{ and } sk = nt + r \ (0 \leq r < n). \tag{5.1'}$$

If G is a cgg with n vertices and e edges, then for every $1 \leq m < n/(2k)$ there is a set M of m lines in $R^2 \setminus V$, whose union meets at least $h(n, e, m)$ edges of G, where $h(n, e, m) = 2m(\binom{k}{2} + t) + \min(2m, \lceil r/k \rceil)$.

Let $G = \langle V, E \rangle$ be a cgg with n vertices and e edges. As in Section 3, let $V = \{v_0, v_1, \ldots, v_{n-1}\}$ and assume that the vertices $v_0, v_1, \ldots, v_{n-1}, v_n = v_0$ appear in this cyclic order on the boundary of conv V. We shall start by proving (5.1′) for $m = 1$. We shall use freely the notions and the notations related to V (i.e., length of an edge b: $d(b) = d(v_iv_j)$, $A = \{a_0, a_1, \ldots, a_{n-1}\}$, $f_G: A \to Z^+$, $l = a_ia_j$, length of the line l: $d(l) = d(a_ia_j)$) that were introduced in Section 3. Denote by d_G the maximum length of an edge of G. By part (ii) of Proposition 3.2, if a_ia_j is a line of length $d \geq d_G$, then the number of edges of G that intersect a_ia_j is precisely $f_G(a_i) + f_G(a_j)$. Therefore, in order to prove (5.1′) for $m = 1$ it suffices to find two segments a_i, a_j, such that $d(a_ia_j) \geq d_G$ and $f_G(a_i) + f_G(a_j) \geq h(n, e, 1)$. In order to do this we need the following lemma. This lemma will be used also to prove (5.1′) for $m > 1$.

Lemma 5.2. *Suppose $lk \leq n$ and let $g: A \to Z$ satisfy*

$$\sum \{g(a): a \in A\} \geq r \geq 0.$$

Then there exists a set $L \subset A$, $|L| = l$, such that $d(ab) \geq k$ for every two distinct elements a, $b \in L$, and

$$\sum \{g(a): a \in L\} \geq \min(\lceil r/k \rceil, l).$$

Proof. Let B be a subset of A of cardinality lk, such that $g(b) \geq g(c)$ for all $b \in B$, $c \in A \setminus B$. Write $B = \{b_0, b_1, \ldots, b_{lk-1}\}$ and assume that the segments $b_0, b_1, \ldots, b_{lk-1}, b_{lk} = b_0$ appear in this cyclic order on the boundary of conv V. For $0 \leq j < k$, define

$$B_j = \{b_{j+vk}: 0 \leq v < l\}.$$

Clearly $|B_j| = l$ and $d(ab) \geq k$ for any two distinct segments a, $b \in B_j$. If $g(b) > 0$ for all $b \in B$, take $L = B_1$. ($\sum \{g(b): b \in B_1\} \geq |B_1| = l$.)

If $g(b) \leq 0$ for some $b \in B$, then $g(c) \leq 0$ for all $c \in A \setminus B$ and therefore

$$r \leq \sum \{g(a): a \in A\} \leq \sum \{g(b): b \in B\} = \sum_{j=0}^{k-1} \sum \{g(b): b \in B_j\}.$$

Hence $\sum \{g(b): b \in B_j\} \geq \lceil r/k \rceil$ for at least one j, and $L = B_j$ satisfies the assertions of the lemma. $\square$

The proof of (5.1′) for $m = 1$ is now almost complete. d_G, the maximum length of an edge of G, is clearly $\geq k-1$. If $d_G = k-1$, then $s = t = r = 0$ and E consists of all edges of length $<k$. In this case $f_G(a_i) = 1 + 2 + \cdots + (k-1) = \binom{k}{2}$ for all $0 \leq i < n$. In particular $f_G(a_0) + f_G(a_k) = k(k-1) = h(n, e, 1)$. If $d_G \geq k$, say $d_G = k + \varepsilon$, then

$$\begin{aligned}\sum_{i=0}^{n-1} f_G(a_i) &= \sum \{d(b): b \in E\} \\ &\geq n(1 + 2 + \cdots + (k-1)) + (s-1)k + (k + \varepsilon) \\ &= n\left(\binom{k}{2} + t\right) + r + \varepsilon.\end{aligned}$$

Substituting 2, $k + \varepsilon$, n, $f_G - \binom{k}{2} - t$ and $r + \varepsilon$ for l, k, n, g and r respectively in Lemma 5.2, we conclude that there are two segments a_i and a_j such that $d(a_i a_j) \geq k + \varepsilon = d_G$ and

$$\begin{aligned}f_G(a_i) + f_G(a_j) &\geq 2\left(\binom{k}{2} + t\right) + \min\left(\left\lceil \frac{r+\varepsilon}{k+\varepsilon} \right\rceil, 2\right) \\ &\geq 2\left(\binom{k}{2} + t\right) + \min(\lceil r/k \rceil, 2) = h(n, e, 1).\end{aligned}$$

This implies the validity of (5.1′) for $m = 1$.

In order to prove (5.1′) for $m > 1$ we define a function $\bar{f}_G: A \to Z^+$, similar to f_G, as follows. Let $b = v_i v_{i+d}$ be an edge of length d of G. (Here $1 \leq d \leq \frac{1}{2}n$. If $d = \frac{1}{2}n$ then $0 \leq i < \frac{1}{2}n$, as in the proof of Theorem 3.1) Recall that

$$W(b) = \{a_i, a_{i+1}, \ldots, a_{i+d-1}\}.$$

Now define

$$\bar{W}(b) = \begin{cases} W(b) & \text{if } d \leqslant k \\ \{a_i, a_{i+1}, \ldots, a_{i+k-1}\} & \text{if } d > k. \end{cases}$$

(Note that $|\bar{W}(b)| = \min(d, k)$.)

For $a \in A$, define

$$\bar{f}_b(a) = \begin{cases} 1 & \text{if } a \in \bar{W}(b), \\ 0 & \text{otherwise,} \end{cases}$$

Finally define, for $a \in A$

$$\bar{f}_G(a) = \sum \{\bar{f}_b(a) : b \in E(G)\} \quad (= |\{b \in E(G) : a \in \bar{W}(b)\}|).$$

We need the following two lemmas.

Lemma 5.3. *Suppose $2mk < n$, $(2m-1)k \geqslant \frac{1}{2}n$, and let B be a subset of A of cardinality $2m$. Write $B = \{b_0, b_1, \ldots, b_{2m-1}\}$ and assume that the segments $b_0, b_1, \ldots, b_{2m-1}$, $b_{2m} = b_0$ appear in this cyclic order on the boundary of conv V. Suppose further that for every two distinct elements b, c of B*

$$d(bc) \geqslant k. \tag{5.2}$$

For $0 \leqslant i < m$ choose a line $l_i = b_i b_{i+m}$ and let $M = \{l_i : 0 \leqslant i < m\}$. Then

$$I(G, \bigcup M) \geqslant \sum_{b \in E(G)} |B \cap \bar{W}(b)| = \sum_{j=1}^{2m} \bar{f}_G(b_j). \tag{5.3}$$

Proof. Note that every two lines l_i, $l_j \in M$ intersect in conv V. Because of (5.2), the contribution of any single edge $b \in E(G)$ to the sum on the right side of (5.3) is either 0 or 1. If the contribution of b is 1, then $\bigcup M$ must intersect b; otherwise, all the segments of B would be on the weak side $W(b)$ of b, and thus the length of b would be at least $(2m-1)k + 1 > \frac{1}{2}n$, which is impossible. Thus, every edge b that contributes 1 to the right side of (5.3) contributes 1 also to the left side, and (5.3) follows. □

Lemma 5.4. *Let M be a family of m lines in $R^2 \setminus V$. Then there exists a family $\bar{M}$ of m lines in $R^2 \setminus V$ such that every two lines of $\bar{M}$ intersect in conv V, and*

$$I(G, \bigcup \bar{M}) \geqslant I(G, \bigcup M).$$

Proof. Clearly we may assume that every line $l \in M$ intersects conv V. If every two lines of M intersect in conv V, there is nothing to prove. Otherwise, M contains two lines l, l' such that $l \cap bd\,\text{conv}\,V = \{p, q\}$, $l' \cap bd\,\text{conv}\,V = \{p', q'\}$, and the points p, q, p', q' are distinct and appear in this cyclic order on bd conv V.

One can easily check that if we modify M by replacing $l = pq$ and $l' = p'q'$ by the lines pp' and qq', then the number of edges that intersect M can only grow, and the number of pairs of lines of M that intersect in conv V increases by at least 1. Repeated application of this procedure leads to the desired family of lines $\bar{M}$. $\square$

In order to complete the proof of Theorem 5.1 we prove (5.1′) for fixed n and e by descending induction on m, for all values of $m > 1$ that satisfy $2mk < n$. Let m be the largest integer that satisfies $2mk < n$, i.e., $m = [(n-1)/2k]$. If $m \leqslant 1$ we have nothing to prove. Otherwise $m \geqslant 2$, and it is easily checked that in this case the maximality of m implies that $(2m-1)k \geqslant \frac{1}{2}n$.

Consider the function $\bar{f}_G\colon A \to Z^+$. Clearly

$$\sum \{\bar{f}_G(a)\colon a \in A\} = \sum \{\bar{f}_b(a)\colon a \in A,\ b \in E(G)\} = \sum \{|\bar{W}(b)|\colon b \in E(G)\}$$

$$\geqslant n(1 + 2 + \cdots + (k-1)) + sk = n\left(\binom{k}{2} + t\right) + r.$$

Substituting $2m$, k, n, $\bar{f}_G - \binom{k}{2} - t$ and r for l, k, n, g and r respectively in Lemma 5.2, we conclude that there is a subset B of A of cardinality $2m$, such that every two distinct elements b, $c \in B$ satisfy $d(bc) \geqslant k$, and

$$\sum \{\bar{f}_G(b)\colon b \in B\} \geqslant 2m\left(\binom{k}{2} + t\right) + \min(2m, \lceil r/k \rceil) = h(n, e, m).$$

This and Lemma 5.3 imply (5.1′) for the maximal possible value of m. For a set M of lines in $R^2 \backslash V$, denote by $A(M)$ the set of segments $a \in A$ that intersect M. The set M produced in Lemma 5.3 clearly satisfies $|A(M)| = 2m$. We continue by descending induction. Assuming we have a set M of $m \geqslant 3$ lines in $R^2 \backslash V$ that satisfies

$$|A(M)| = 2m \quad \text{and} \quad I(G, \bigcup M) \geqslant 2mc + \min(2m, \lceil r/k \rceil), \tag{5.4}$$

where $c = \binom{k}{2} + t$, we shall produce a set M' of $m-1$ lines that satisfies

$$\begin{aligned} &|A(M')| = 2m - 2, \qquad A(M') \subset A(M) \quad \text{and} \\ &I(G, \bigcup M') \geqslant 2(m-1)c + \min(2(m-1), \lceil r/k \rceil). \end{aligned} \tag{5.5}$$

By Lemma 5.4 we may assume that every two lines in M intersect in conv V. For every line $l \in M$ let E_l be the set of edges of G that intersect l and do not intersect any other line in M. If $|E_l| \leqslant 2c$ for some $l \in M$ then $M' = M \backslash \{l\}$ clearly satisfies (5.5). Thus we may assume that $|E_l| > 2c$ for all $l \in M$. If $|E_l| \geqslant 2c + 2$ for all $l \in M$ except, possibly, for one line l_0, then $M' = M \backslash \{l_0\}$ clearly satisfies (5.5). Thus we may assume that there are $g \geqslant 2$ lines l of M for which $|E_l| = 2c + 1$ (these will be referred to as lines of the first kind), and that $|E_l| \geqslant 2c + 2$ for all other lines l of M. We consider two possible cases.

Case 1. For every two distinct lines l, l' of the first kind there exists at least one edge of G that intersects both l and l', but no other line in M. In this case

$$\begin{aligned} I(G, \bigcup M) &\geqslant \sum \{|E_l| : l \in M\} + \binom{g}{2} \\ &\geqslant g(2c+1) + (m-g)(2c+2) + \binom{g}{2} \\ &= m(2c+2) + \binom{g}{2} - g \geqslant m(2c+2) - 1. \end{aligned}$$

Let $M' = M \setminus \{l\}$, where l is a line of the first kind. Then

$$I(G, \bigcup M') = I(G, \bigcup M) - (2c+1) \geqslant 2(m-1)c + 2(m-1),$$

which implies (5.5).

Case 2. There exist two distinct lines $l = a_i a_j$ and $l' = a_{i'} a_{j'}$ of the first kind, such that every edge of G that intersects both l and l' meets at least one additional line of M.

By (5.4), the four segments a_i, $a_{i'}$, a_j, $a_{j'}$ are distinct, and they appear in this cyclic order on *bd* conv V, since we assume that l and l' intersect in conv V. Divide E_l into two disjoint subsets $E_{l,i}$ and $E_{l,j}$ as follows: If $b \in E_l$, then $b \in E_{l,i}$ [$b \in E_{l,j}$] iff b and a_i [resp. a_j] lie on the same side of the line l'. (Remember that if $b \in E_l$ then b does not meet l'.) Similarly divide $E_{l'}$ into $E_{l',i'}$ and $E_{l',j'}$. Since $|E_l| = 2c+1$, exactly one of the numbers $|E_{l,i}|$, $|E_{l,j}|$ is $\geqslant c+1$. Assume, w.l.o.g., that $|E_{l,i}| \geqslant c+1$. Similarly we may assume that $|E_{l',i'}| \geqslant c+1$.

Now define $M' = M \setminus \{l, l'\} \cup \{a_i a_{i'}\}$. Since we are in Case 2, every edge of G that meets M and is not in $E_l \cup E_{l'}$ intersects M'. One can also easily verify that every edge in $E_{l,i} \cup E_{l',i'}$ intersects the new line $a_i a_{i'}$. Therefore

$$I(G, \bigcup M') = I(G, \bigcup M) - |E_{l,j}| - |E_{l',j'}| \geqslant I(G, \bigcup M) - 2c,$$

and (5.5) follows.

This completes the proof of (5.1′) for $m > 1$, and establishes Theorem 5.1. □

6. Concluding remarks

We would like to mention some natural variants of the problems considered in this paper.

(a) One can regard the edges of a gg $G = (V, E)$ as *closed* line segments. Seeking lines that touch many edges of G, under this definition, we may obviously restrict our attention to lines determined by pairs of vertices of G.

Moreover, in this case the minimum of $I(G, m)$ over all gg's with n vertices and e edges does not decrease if we drop the condition that the vertices of G be in general position. (We do not know whether under the definitions of Section 1, the function $I(n, e, m)$ is affected if V is not required to be in general position.)

The methods used in this paper can be easily adapted to deal with the 'closed edge' analogues of the functions $I(n, e, m)$, $I_c(n, e, m)$. The results are similar to those obtained in this paper, with the function h (see (1.3)) replaced by h', as defined below.

Suppose $n \geqslant 1$, $0 \leqslant e \leqslant \binom{n}{2}$, $1 \leqslant m < \frac{1}{2}n$, and

$$\begin{aligned} n &= 2mv - \rho \qquad (0 \leqslant \rho < 2m), \\ e &= n(k-1) + s \qquad (0 \leqslant s < n), \\ s(k+1) &= nt + r \qquad (0 \leqslant r < n). \end{aligned}$$

If $k < v - 1$, then $h'(n, e, m) = m(k+2)(k-1) + 2mt + \min(2m, \lceil r/(k+1) \rceil)$.
If $k \geqslant v - 1$, then $h'(n, e, m) = e - \rho\binom{v-2}{2} - (2m - \rho)\binom{v-1}{2}$.

(b) Instead of (1.1), one can ask for the minimum of $I(G, m)$, where G ranges over some restricted class of gg's. For instance, one could regard all gg's (or cgg's) G that are isomorphic (as abstract graphs) to a given abstract graph Γ.

(c) One can investigate properties of $I(G, m)$ as a random variable over some class of random gg's on a fixed set of vertices.

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